A FORMULA FOR THE MULTIPLICITY OF THE MULTI-GRADED REES ALGEBRA

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1. INTRODUCTION

Our aim in this paper is to obtain a formula for the multiplicity of the maximal homogeneous ideal of the multi-graded extended Rees algebra. This formula generalizes the one obtained in [8] by Katz and Verma for the ordinary extended Rees algebra.

The extended Rees algebra was introduced by Rees in [10] and has been of interest in the past decade. The Rees algebra which is a subring of the extended Rees algebra have been extensively studied. It is natural to expect these two algebras to share some ring-theoretic properties. For example, if I is an ideal of positive height in a Cohen-Macaulay local ring R, then extended Rees algebra of I is Cohen-Macaulay whenever the Rees algebra is [6].

Recently, the multi-graded Rees algebra has been investigated ([7], [5], [14], [17], [2]). In particular, the multiplicity of the maximal homogeneous ideal of the multi-graded Rees algebra was obtained independently by Verma ([17, Theorem 1.4]) and Herrmann et. al. ([5, Corollary 4.7]). Since the multi-graded Rees algebra is a subring of the multi-graded extended Rees algebra, it is natural to study the multi-graded extended Rees algebra. In this paper we concentrate on the multiplicity of the maximal homogeneous ideal of this ring.

Throughout this paper $(R, \mathfrak{m})$ will denote a Noetherian local ring of positive dimension d with infinite residue field. Let $I_1, \ldots, I_g$ be ideals of positive height in R and let $t_1, \ldots, t_g$ be indeterminates. The *multi-graded extended Rees algebra* of R with respect to

$I_1, \ldots, I_g$ is the graded ring $\bigoplus_{(r_1,\ldots,r_g)\in\mathbb{Z}^g}(I_1t_1)^{r_1}\cdots(I_gt_g)^{r_g}$ and will be denoted by $\mathcal{B}(\mathbf{I})$. Here $I_i^{r_i} = R$ if $r_i \leq 0$. Let $\mathcal{N}(\mathbf{I}) = (I_1t_1, \ldots, I_gt_g, \mathfrak{m}, t_1^{-1}, \ldots, t_g^{-1})$. The *multi-graded Rees algebra* of R with respect to $I_1, \ldots, I_g$ is the graded ring $\bigoplus_{(r_1,\ldots,r_g)\in\mathbb{N}^g}(I_1t_1)^{r_1}\cdots(I_gt_g)^{r_g}$ and will be denoted by $\mathcal{R}(\mathbf{I})$. Let $\mathcal{M}(\mathbf{I})$ be the maximal homogeneous ideal of $\mathcal{R}(\mathbf{I})$. When $g = 1$, we say $\mathcal{B}(I)$ is the extended Rees algebra and $\mathcal{R}(I)$ is the Rees algebra.

To state our main result we need to define mixed multiplicities. Let I_1 be an $\mathfrak{m}$-primary ideal and let $I_2, \ldots, I_g$ be ideals of positive height in $(R, \mathfrak{m})$. Then for $r_1, \ldots, r_g$ large, $\ell_R(I_1^{r_1}I_2^{r_2}\cdots I_g^{r_g}/I_1^{r_1+1}I_2^{r_2}\cdots I_g^{r_g})$ is a polynomial of degree $d-1$ and the terms of degree $d-1$ are captured in the following sum

$$\sum_{q_1+\cdots+q_g=d-1} e(I_1^{[q_1+1]}|I_2^{[q_2]}|\cdots|I_g^{[q_g]})\binom{r_1+q_1}{q_1}\cdots\binom{r_g+q_g}{q_g}.$$

Here $e(I_1^{[q_1+1]}|\cdots|I_g^{[q_g]})$ are positive integers and they are called the *mixed multiplicities* of the set of ideals $(I_1, \ldots, I_g)$ [15].

We state the multiplicity formula for the multi-graded Rees algebra. In this paper, if $(R, \mathfrak{m})$ is a local ring, then $e(I)$ will denote the multiplicity of an $\mathfrak{m}$-primary ideal in R and $e(R)$ will denote the multiplicity of the maximal ideal $\mathfrak{m}$ of R.

Theorem 1.1. ([17, Theorem 1.4], [5, Corollary 4.7]) *Let $I_1, \ldots, I_g \subseteq \mathfrak{m}$ be ideals of positive height in R. Then*

$$e(\mathcal{R}(\mathbf{I})_{\mathcal{M}(\mathbf{I})}) = \sum_{q+q_1+\cdots+q_g=d-1} e(\mathfrak{m}^{[q+1]}|I_1^{[q_1]}|\cdots|I_g^{[q_g]}).$$

The main result of this paper is:

Theorem 1.2. *Let $I_1, \ldots, I_g \subseteq \mathfrak{m}$ be ideals of positive height in R. Put $L = \mathfrak{m}^2 + I_1 + \cdots + I_g$. Then*

$$e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) = \frac{1}{2^d}\left[\sum_{t=0}^{g}\sum_{\substack{q+q_1+\cdots+q_t=d-1\\ 1\leq i_1<\cdots<i_t\leq g}} 2^{q_1+\cdots+q_t} e(L^{[q+1]}|I_{i_1}^{[q_1]}|\cdots|I_{i_t}^{[q_t]})\right].$$

To attain our goal, we need to express mixed multiplicities of certain homogeneous ideals in the extended Rees ring in terms of mixed multiplicities of ideals in the ring R

(Proposition 3.3). We recover the multiplicity formula obtained by Katz and Verma for the extended Rees algebra (see Corollary 3.6).

We now describe the organization of this paper. In Section three we prove our main result. Section two is devoted to develop the necessary preliminary results. We end this paper by explicitly stating the multiplicity formula for $\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}$ when $d = 1, 2$.

2. PRELIMINARIES

In this section we prove a few basic facts.

Lemma 2.1. *Let r, s and n be non-negative integers. Then*

$$\sum_{i=0}^{n} \binom{i+r}{r}\binom{n-i+s}{s} = \binom{n+r+s+1}{r+s+1}.$$

Proof. Apply induction on $n + r$. □

Lemma 2.2. *Let s be a positive integer and let r and n be indeterminates. Then*

$$\binom{nr+s}{s} = n^s\binom{r+s}{s} + f_1(n)\binom{r+s-1}{s-1} + \cdots + f_s(n)$$

where $f_1(n), \ldots, f_s(n) \in \mathbb{Q}(n)$.

Proof. Since $\{\binom{r+i}{i}\}_{i\in\mathbb{N}}$ form a basis of $\mathbb{Q}(n)[r]$ as a vector space over $\mathbb{Q}(n)$, we can write

$$\binom{nr+s}{s} = \sum_{i=0}^{s} f_i(n)\binom{r+s-i}{s-i}$$

where $f_i(n) \in \mathbb{Q}(n)$, $0 \le i \le s$. Comparing the coefficient of r^s we get $n^s = f_0(n)$. □

Lemma 2.3. *Let K be an $\mathfrak{m}$-primary ideal and let $J_1, \ldots, J_g \subseteq \mathfrak{m}$ be ideals of positive height in R. Then for $r_1, \ldots, r_g$ large,*

$$F(r_1, \ldots, r_g) = \ell\left(\frac{J_1^{r_1} \ldots J_g^{r_g}}{KJ_1^{r_1} \ldots J_g^{r_g}}\right)$$

is a polynomial of total degree at most $d-1$ in $r_1, \ldots, r_g$.

Proof. For all large values of $r_1,\dots,r_g$, $F(r_1,\dots,r_g)$ is a polynomial say $P(r_1,\dots,r_g)\in\mathbb{Q}[r_1,\dots,r_g]$ [5, Theorem 4.1]. Since the monomials of highest degree in $r_1,\dots,r_g$ have non-negative coefficients, the total degree of $P(r_1,\dots,r_g)$ is equal to the degree of $P(r,\dots,r)$ and

$$\begin{aligned}
& \deg\ P(r,\dots,r) & \\
= & \dim\ \left(\frac{R[(J_1\cdots J_g)t]}{KR[(J_1\cdots J_g)t]}\right)-1 & \text{[by [5, Theorem 4.1, Lemma 1.1]]}\\
\leq & \dim\ R[(J_1\cdots J_g)t]-2 & \text{[by [11, Theorem 1.5]]}\\
= & d-1 & \text{[by [16, Corollary 1.6]].}
\end{aligned}$$

□

Lemma 2.4. *Let J be an $\mathfrak{m}$-primary ideal and let $I\subseteq\mathfrak{m}$ be an ideal of positive height in R. Put $T=\mathcal{B}(I)$, $M=(t^{-1},J,It)$, $K=J^2+I$ and $H=J+I$. Then for $0\leq j<r$,*

$$\begin{aligned}
& M^{2(r-j)}\\
= & \bigoplus_{i=0}^{\infty}Rt^{-(2r+i)}\bigoplus\bigoplus_{i=1}^{r-1}K^{i-j}t^{-(2r-2i)}\bigoplus\bigoplus_{i=0}^{j-1}Rt^{-(2r-2i-1)}\bigoplus\bigoplus_{i=j}^{r-1}HK^{i-j}t^{-(2r-2i-1)}\bigoplus K^{r-j}\\
& \bigoplus\bigoplus_{i=0}^{\infty}(It)^{(2r+i)}\bigoplus\bigoplus_{i=1}^{r-1}K^{i-j}(It)^{(2r-2i)}\bigoplus\bigoplus_{i=0}^{j-1}(It)^{2r-2i-1}\bigoplus\bigoplus_{i=j}^{r-1}HK^{i-j}(It)^{2r-2i-1}.
\end{aligned}$$

Proof. An expression for M^n has been obtained in [8, Lemma 3.1]. By arranging and re-indexing the terms we get the expression in the above form. □

3. THE MAIN THEOREM

In this section we prove our main result. One of the main ingredients is Proposition 3.3.

Notation: Put $\mathcal{B}_0:=R$, $\mathcal{N}_0:=\mathfrak{m}$, $L=\mathfrak{m}^2+I_1+\cdots+I_g$ and $I_{g+1}=(0)$. For $j=1,\dots,g$ we inductively define $\mathcal{B}_j=\mathcal{B}_{j-1}[I_jt_j,t_j^{-1}]$, $\mathcal{N}_j=(t_j^{-1},\mathcal{N}_{j-1},I_j\mathcal{B}_{j-1}t_j)$ and $L_j=\mathcal{N}_j^2+(I_{j+1}+\cdots+I_g)\mathcal{B}_j$.

Lemma 3.1. *Let $(R,\mathfrak{m})$ be a local ring of positive dimension d. Let $I_1,\dots I_g$ be ideals of positive height in R. Then* $\dim\ \mathcal{B}(\mathbf{I})=\dim\ R+g$.

Proof. Since $\mathcal{B}_g = \mathcal{B}_{g-1}[I_g t_g, t_g^{-1}]$, it is enough to show that dim $\mathcal{B}_1 =$ dim $R+1$ and that the ideal $I_2\mathcal{B}_1$ has positive height. Clearly $\mathcal{B}_1/(t_1^{-1})$ is the associated graded ring $G = \oplus_{n\geq 0} I_1^n/I_1^{n+1}$ and dim $G = \dim R$. Hence, dim $\mathcal{B}(I) =$ dim $G+1 =$ dim $R+1$. If the height of $I_2\mathcal{B}_1$ is zero, then it is contained in some minimal prime of $\mathcal{B}_1$. By a result of Valla, [16, cf. Proposition 1.1 (iii)], $I_2 = I_2\mathcal{B}_1 \cap R$ is contained in some minimal prime of R which leads to a contradiction. □

Lemma 3.2. *Let I be an ideal of positive height in $(R, \mathfrak{m})$. Let $\mathcal{I}_1$ be an $\mathcal{N}(I)$-primary homogeneous ideal and let $\mathcal{I}_2, \ldots, \mathcal{I}_g$ be homogeneous ideals of positive height in $\mathcal{B}(I)$. Then for all non-negative integers $q_1, \ldots, q_g$ with $q_1 + \cdots + q_g = d-1$ we have*

$$e((\mathcal{I}_{1\mathcal{N}(I)})^{[q_1+1]}|(\mathcal{I}_{2\mathcal{N}(I)})^{[q_2]}|\ldots|(\mathcal{I}_{g\mathcal{N}(I)})^{[q_g]}) = e(\mathcal{I}_1^{[q_1+1]}|\mathcal{I}_2^{[q_2]}|\ldots|\mathcal{I}_g^{[q_g]}).$$

Proof. Since $\mathcal{N}(I)$ is the maximal homogeneous ideal of $\mathcal{B}(I)$ and $\mathcal{I}_2, \ldots, \mathcal{I}_g$ are homogeneous ideals the following isomorphism holds true:

$$\frac{\mathcal{I}_1^{r_1}\mathcal{I}_2^{r_2}\ldots\mathcal{I}_g^{r_g}}{\mathcal{I}_1^{r_1+1}\mathcal{I}_2^{r_2}\ldots\mathcal{I}_g^{r_g}} \simeq \frac{(\mathcal{I}_1^{r_1}\mathcal{I}_2^{r_2}\ldots\mathcal{I}_g^{r_g})_{\mathcal{N}(I)}}{(\mathcal{I}_1^{r_1+1}\mathcal{I}_2^{r_2}\ldots\mathcal{I}_g^{r_g})_{\mathcal{N}(I)}}.$$

□

Notation: Let I_1 be an $\mathfrak{m}$-primary ideal and let I_2 be an ideal of positive height in $(R, \mathfrak{m})$. For $g = 2$, we will use the notation

$$\begin{aligned} e_q(I_1|I_2) &:= e(I_1^{[d-q]}|I_2^{[q]}) && q = 0, \ldots, d-1, \\ e_d(I_1|I_2) &:= 0. \end{aligned}$$

Proposition 3.3. *Let J be an $\mathfrak{m}$-primary ideal and let $I, I_1 \subseteq \mathfrak{m}$ be ideals of positive height in R. Let $J_1 \subseteq J + I$ be any ideal of R. Put $T = \mathcal{B}(I)$, $M = (t^{-1}, J, It)$ and $K = J^2 + I$. Then for all $q = 0, \ldots, d$*

$$e_q(M^2 + J_1T|I_1T) = 2\left[e_q(K+J_1|I_1) + \sum_{q_0+q_1+q=d-1} 2^{q_1} e((K+J_1)^{[q_0+1]}|I^{[q_1]}|I_1{}^{[q]})\right].$$

The following lemma is well known and easy to see, but nevertheless we mention it for the sake of completion.

Lemma 3.4. *Let I be an $\mathfrak{m}$-primary ideal and let J be any ideal of positive height in a local ring of positive dimension d. Then for all $q = 0, \dots d-1$,*

(1) $e_q(I^r|J^s) = r^{d-q}s^q e_q(I|J)$.
(2) $e_0(I|J) = e(I)$.

Proof. It is easy to see that for $r, s \gg 0$

$$\ell\left(\frac{(I^n)^r(J^m)^s}{(I^n)^{r+1}(J^m)^s}\right) = \sum_{i=0}^{n-1} \ell\left(\frac{I^{nr+i}J^{ms}}{(I^{nr+i+1}J^{ms}}\right).$$

Comparing the coefficient of $r^{d-1-q}s^q$ on both sides we get (1). The second result was proved by D. Katz and J. Verma in [8, Lemma 2.2]. $\square$

Corollary 3.5. *Let J be an $\mathfrak{m}$-primary ideal and let $I, I_1 \subseteq \mathfrak{m}$ be ideals of positive height in R. Put $T = \mathcal{B}(I)$, $M = (t^{-1}, J, It)$. Then for all $q = 0, \dots, d$*

$$e_q(M|I_1T) \quad = \quad \frac{1}{2^{d-q}}\left[e_q(J^2+I|I_1) + \sum_{q_0+q_1+q=d-1} 2^{q_1} e((J^2+I)^{[q_0+1]}|I^{[q_1]}|{I_1}^{[q]})\right].$$

Proof. Since $\dim\ \mathcal{B}(I) = d+1$, $e_q(M^2|I_1) = 2^{d+1-q}e_q(M|I_1T)$ by Lemma 3.4. Put $J_1 = (0)$ in Proposition 3.3. Then for all $q = 0, \dots, d$

$$e_q(M^2|I_1) = 2\left[e_q(J^2+I|I_1) + \sum_{q_0+q_1+q=d-1} 2^{q_1} e((J^2+I)^{[q_0+1]}|I^{[q_1]}|{I_1}^{[q]})\right].$$

$\square$

Proof of Proposition 3.3: First note that by Lemma 3.2,

$$\frac{((M^2+J_1T)^r(I_1T)^s)_{\mathcal{N}(I)}}{((M^2+J_1T)^{r+1}(I_1T)^s)_{\mathcal{N}(I)}} = \frac{(M^2+J_1T)^r(I_1T)^s}{(M^2+J_1T)^{r+1}(I_1T)^s}.$$

Since $\dim\ \mathcal{B}(I) = d+1$, for $r, s \gg 0$, $\ell(((M^2+J_1T)^r(I_1T)^s)_{\mathcal{N}(I)}/((M^2+J_1T)^{r+1}(I_1T)^s)_{\mathcal{N}(I)})$ is a polynomial of total degree d in r and s [1] and can be written in the form

$$\ell\left(\frac{(M^2+J_1T)^r(I_1T)^s_{\mathcal{N}(I)}}{(M^2+J_1T)^{r+1}(I_1T)^s_{\mathcal{N}(I)}}\right) = \sum_{q=0}^{d} e_q(M^2+J_1T|I_1T)\binom{r+d-q}{d-q}\binom{s+q}{q} + \cdots. \quad (1)$$

But $(M^2+J_1T)^rI_1T^s$ is a graded ideal. and for all non-negative integers r and s, the module $(M^2+J_1T)^r(I_1T)^s/(M^2+J_1T)^{r+1}(I_1T)^s$ can be expressed as a (finite) direct sum of R- modules which have finite length.

Put $H = I + J$. Notice that $(I_1T)^s = {I_1}^sT$. It follows from Lemma 2.4 that

$$
\begin{aligned}
&(M^2 + J_1T)^r(I_1T)^s \\
= \; & \bigoplus_{i=0}^{\infty} {I_1}^s t^{-(2r+i)} \bigoplus \bigoplus_{i=1}^{r-1} {I_1}^s(K+J_1)^i t^{-(2r-2i)} \bigoplus \bigoplus_{i=0}^{r-1} {I_1}^s H(K+J_1)^i t^{-(2r-2i-1)} \bigoplus {I_1}^s(K+J_1)^r \\
& \bigoplus \bigoplus_{i=0}^{\infty} {I_1}^s(Jt)^{2r+i} \bigoplus \bigoplus_{i=1}^{r-1} {I_1}^s(K+J_1)^i(Jt)^{2r-2i} \bigoplus \bigoplus_{i=0}^{r-1} {I_1}^s H(K+J_1)^i(Jt)^{2r-2i-1}
\end{aligned}
$$

for all $r \geq 1$ and for all $s \geq 0$. Therefore

$$
\begin{aligned}
&\frac{(M^2 + J_1T)^r(I_1T)^s}{(M^2 + J_1T)^{r+1}(I_1T)^s} \\
= \; & \left[\frac{{I_1}^s}{(K+J_1){I_1}^s} \bigoplus \frac{{I_1}^s}{H{I_1}^s}\right] \bigoplus \bigoplus_{i=1}^{r-1} \frac{(K+J_1)^i{I_1}^s}{(K+J_1)^{i+1}{I_1}^s} \bigoplus \bigoplus_{i=0}^{r-1} \frac{H(K+J_1)^i{I_1}^s}{H(K+J_1)^{i+1}{I_1}^s} \\
& \bigoplus \frac{(K+J_1)^r{I_1}^s}{(K+J_1)^{r+1}{I_1}^s} \bigoplus \left[\frac{J^{2r}{I_1}^s}{J^{2r}(K+J_1){I_1}^s} \bigoplus \frac{J^{2r+1}{I_1}^s}{J^{2r+1}H{I_1}^s}\right] \bigoplus \bigoplus_{i=1}^{r-1} \frac{(K+J_1)^iJ^{2r-2i}{I_1}^s}{(K+J_1)^{i+1}J^{2r-2i}{I_1}^s} \\
& \bigoplus \bigoplus_{i=0}^{r-1} \frac{H(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{H(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}.
\end{aligned}
$$

For large exponents the length of the modules appearing in the above sum are polynomials. We are interested only in those modules whose length will contribute to the terms of total degree d in r and s. In what follows, we will denote by $\cdots$ a function in r and s of total degree less than d. For large exponents $\ell({I_1}^s/H{I_1}^s)$, $\ell(J^{2r+1}{I_1}^s/J^{2r+1}H{I_1}^s)$ and $\ell((K+J_1)^r{I_1}^s/(K+J_1)^{r+1}{I_1}^s)$ are polynomials of total degree atmost $d-1$ [Lemma 2.3]. Consider,

$$
\begin{aligned}
\ell\left(\frac{H(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{H(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}\right) &= \ell\left(\frac{(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}\right) \\
&+ \left[\ell\left(\frac{(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}{H(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}\right) - \ell\left(\frac{(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{H(K+J_1)^iJ^{2r-2i-1}{I_1}^s}\right)\right].
\end{aligned}
$$

By Lemma 2.3, for large exponents the length of both the modules appearing in the square bracket are polynomials of total degree at most $d-1$. Moreover, the coefficients of all the monomials of highest degree appearing in both the polynomials are the same. Hence, their difference is a polynomial of total degree at most $d-2$. Thus

$$
\begin{aligned}
\ell\left(\frac{H(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{H(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}\right) &= \ell\left(\frac{(K+J_1)^iJ^{2r-2i-1}{I_1}^s}{(K+J_1)^{i+1}J^{2r-2i-1}{I_1}^s}\right) + \cdots \\
&= \ell\left(\frac{(K+J_1)^iJ^{2r-2i}{I_1}^s}{(K+J_1)^{i+1}J^{2r-2i}{I_1}^s}\right) + \cdots .
\end{aligned}
$$

for large r and s. Similarly one can show

$$\ell\left(\frac{H(K+J_1)^i {I_1}^s}{H(K+J_1)^{i+1} {I_1}^s}\right) = \ell\left(\frac{(K+J_1)^i {I_1}^s}{(K+J_1)^{i+1} {I_1}^s}\right) + \cdots .$$

Thus considering the relevant terms we get

$$\begin{aligned}
&\ell\left(\frac{(M^2+J_1T)^r {I_1}^s T}{(M^2+J_1T)^{r+1} {I_1}^s T}\right) \\
&= 2\left[\sum_{i=0}^{r-1} \ell\left(\frac{(K+J_1)^i {I_1}^s}{(K+J_1)^{i+1} {I_1}^s}\right) + \sum_{i=0}^{r-1} \ell\left(\frac{(K+J_1)^i J^{2r-2i} {I_1}^s}{(K+J_1)^{i+1} J^{2r-2i} {I_1}^s}\right)\right] + \cdots . \qquad (2)
\end{aligned}$$

For large exponents the terms which appear in the above sums are polynomials of degree $d-1$. Without loss of generality, we can assume that they are polynomials for all exponents since the multiplicity formula will not be altered. Consider

$$\begin{aligned}
&\sum_{i=0}^{r-1} \ell\left(\frac{(K+J_1)^i J^{2r-2i} {I_1}^s}{(K+J_1)^{i+1} J^{2r-2i} {I_1}^s}\right) \\
&= \sum_{i=0}^{r-1} \sum_{q_0+q_1+q=d-1} e((K+J_1)^{[q_0+1]}|J^{[q_1]}|{I_1}^{[q]}) \binom{i+q_0}{q_0}\binom{2r-2i+q_1}{q_1}\binom{s+q}{q} + \cdots \\
&= 2^{q_1} \sum_{q_0+q_1+q=d-1} e((K+J_1)^{[q_0+1]}|J^{[q_1]}|{I_1}^{[q]}) \binom{r+d-q}{d-q}\binom{s+q}{q} + \cdots \\
&\qquad \text{[by Lemma 2.2 and Lemma 2.1].} \qquad (3)
\end{aligned}$$

Similarly, one can show that

$$\sum_{i=0}^{r} \ell\left(\frac{(K+J_1)^i {I_1}^s}{(K+J_1)^{i+1} {I_1}^s}\right) = \sum_{q=0}^{d-1} e_q(K+J_1|I_1)\binom{r+d-q}{d-q}\binom{s+q}{q} + \cdots . \qquad (4)$$

Substitute (3) and (4) in (2). By comparing the coefficient of $r^{d-q}s^q$ $(0 \le q \le d-1)$ in (2) and (1) we get the desired result. $\square$

Corollary 3.6. [8, Theorem 3.4] *Let I be an ideal of positive height and let J be an $\mathfrak{m}$-primary in R. Let $T = \mathcal{B}(I)$ and let $M = (t^{-1}, J, It)$. Then*

$$e(M) = \frac{1}{2^d}\left[e(J^2+I) + \sum_{q=0}^{d-1} 2^q e_q(J^2+I|I)\right]$$

Proof. Put $s = 0$ in the above proof. $\square$

Corollary 3.7. *Let I be an $\mathfrak{m}$-primary ideal and let $J, I_1, \dots, I_n \subseteq \mathfrak{m}$ be ideals of positive height in R. Let $J_1 \subseteq I + J$ be any ideal of R. Put $T = \mathcal{B}(J)$, $M = (t^{-1}, I, Jt)$ and $K = I^2 + J$. Then for all non-negative integers $q, q_1, \dots, q_n$ satisfying $q + q_1 + \dots + q_n = d - 1$,*

$$\begin{aligned} & e((M^2 + J_1T)^{[q_0+2]}|(I_1T)^{[q_1]}|\cdots|(I_nT)^{[q_n]}) \\ = \; & 2\left[e((K+J_1)^{[q_0+1]}|I_1^{[q_1]}|\cdots|I_n^{[q_n]}) + \sum_{k+l=q_0} 2^l e((K+J_1)^{[k+1]}|J^{[l]}|I_1^{[q_1]}|\cdots|I_n^{[q_n]})\right]. \end{aligned}$$

Proof. The proof follows by replacing $(IT)^s$ by $(I_1T)^{s_1}\dots(I_nT)^{s_n}$ in (2) of Proposition 3.3 and by using arguments similar to those in Proposition 3.3. $\square$

Lemma 3.8. *Let $I_1, \dots, I_g \subseteq \mathfrak{m}$ be ideals of positive height in R. Put $I_{g+1} = (0)$. Let $1 \leq j \leq g$. Then for all for all $j = 1, \dots, g$ and for all non-negative integers $q_0, q_{j+1}, \dots, q_g$ satisfying $q_0 + q_{j+1} + \dots + q_g = d - 1 - j$,*

$$\begin{aligned} & e(L_j^{[q_0+j+1]}|I_{j+1}\mathcal{B}_j^{[q_{j+1}]}|\cdots|I_g\mathcal{B}_j^{[q_g]}) \\ = \; & 2^j \sum_{t=0}^{j} \sum_{\substack{q+q_1+\dots+q_t=q_0 \\ 1\leq i_1<\dots<i_t\leq j}} 2^{q_0-q} e(L^{[q+1]}|I_{i_1}^{[q_1]}|\cdots|I_{i_t}^{[q_t]}|I_{j+1}^{[q_{j+1}]}|\cdots|I_g^{[q_g]}). \end{aligned}$$

Proof. Notice that $L_j = \mathcal{N}_j^2 + (I_{j+1} + \dots + I_g)\mathcal{B}_j$, $1 \leq j \leq g$.

We induct on j. Let $j = 1$. In Corollary 3.7, put $n = g$, $T = \mathcal{B}_1$, $M = \mathcal{N}_1$, $J = I_1$, $I = \mathfrak{m}$, $J_1 = I_2 + \dots + I_g$, and replace the set of ideals $\{I_1, \dots, I_n\}$ by the set of ideals $\{I_2, \dots, I_g\}$. Also put $l = q_1$ and $k = q$.

Suppose $j > 1$. In Corollary 3.7 put $T = \mathcal{B}_j$, $M = \mathcal{N}_j$ and $R = \mathcal{C}_{j-1} := (\mathcal{B}_{j-1})_{N_j}$ $J = I_j$, $I = \mathcal{N}_{j-1}$, $J_1 = I_{j+1} + \dots + I_g$ and replace the set of ideals $\{I_1, \dots, I_n\}$ by the set of ideals $\{I_{j+1}, \dots, I_g\}$. Also put $l = q_j$ and $k = q$. Then

$$
\begin{aligned}
& e(L_j^{[q_0+j+1]}|I_{j+1}\mathcal{B}_j^{[q_{j+1}]}|\cdots|I_g\mathcal{B}_j^{[q_g]}) \\
= \;& 2\left[e(L_{j-1}^{[q_0+j]}|I_{j+1}\mathcal{C}_{j-1}^{[q_{j+1}]}|\cdots|I_g\mathcal{C}_{j-1}^{[q_g]})\right. \\
& \left. + \sum_{q+q_j=q_0+j-1} 2^{q_j} e(L_{j-1}^{[q+1]}|I_j\mathcal{C}_{j-1}^{[q_j]}|I_{j+1}\mathcal{C}_{j-1}^{[q_{j+1}]}|\cdots|I_n\mathcal{C}_{j-1}^{[q_n]})\right] \qquad \text{[by Corollary 3.7]} \\
= \;& 2\left[e(L_{j-1}^{[q_0+j]}|I_{j+1}\mathcal{B}_{j-1}^{[q_{j+1}]}|\cdots|I_g\mathcal{B}_{j-1}^{[q_g]})\right. \\
& \left. + \sum_{q+q_j=q_0+j-1} 2^{q_j} e(L_{j-1}^{[q+1]}|I_j\mathcal{B}_{j-1}^{[q_j]}|I_{j+1}\mathcal{B}_{j-1}^{[q_{j+1}]}|\cdots|I_g\mathcal{B}_{j-1}^{[q_g]})\right] \qquad \text{[by Lemma 3.2].}
\end{aligned}
$$

By induction hypothesis, each term in the above bracket can be expressed as a sum of mixed multiplicities of ideals in the ring R. Combining these terms in a nice way we get the desired result. $\square$

Proof of Theorem 1.2 Since $\mathcal{N}(\mathbf{I})$ is a maximal ideal in $\mathcal{B}(\mathbf{I})$,

$$\ell\left(\frac{\mathcal{N}(\mathbf{I})^r\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}}{\mathcal{N}(\mathbf{I})^{r+1}\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}}\right) = \ell\left(\frac{\mathcal{N}(\mathbf{I})^r}{\mathcal{N}(\mathbf{I})^{r+1}}\right).$$

By Lemma 3.1, $\dim \mathcal{B}_g = d+g$. Hence $e(\mathcal{N}(\mathbf{I})^2) = 2^{d+g}e(\mathcal{N}(\mathbf{I}))$. Also $e(L_g^{[d+g]}) = e(L_g) = e(\mathcal{N}(\mathbf{I})^2)$. Put $j = g$ in Lemma 3.8. This completes the proof of the theorem. $\square$

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a local ring of dimension one. Let $I_1, \ldots, I_g$ be ideals of positive height. Put $L = \mathfrak{m}^2 + I_1 + \cdots + I_g$. Then*

$$e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) = 2^{g-1}e(L).$$

Proof. Put $d = 1$ in Theorem 1.2. Then

$$e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) = \frac{1}{2}\left[\sum_{t=0}^{g}\sum_{1\le i_1<\cdots<i_t\le g} e(L)\right] = \frac{e(L)}{2}\left[\sum_{t=0}^{g}\binom{g}{t}\right] = 2^{g-1}e(L).$$

$\square$

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a local ring of dimension two. Let $I_1, \ldots, I_g$ be ideals of positive height. Put $L = \mathfrak{m}^2 + I_1 + \ldots + I_g$. Then*

$$e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) = 2^{g-2}\left[e(L) + \sum_{j=1}^{g} e_1(L|I_j)\right].$$

Proof. Put $d = 2$ in Theorem 1.2. Then

$$\begin{aligned}
e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) &= \frac{1}{4}\left[\sum_{t=0}^{g}\sum_{\substack{q+q_1+\cdots+q_t=1\\1\leq i_1<\cdots<i_t\leq g}} 2^{q_1+\cdots+q_t} e(L^{[q+1]}|I_{i_1}^{[q_1]}|\cdots|I_{i_t}^{[q_t]})\right]\\
&= \frac{1}{4}\left[\sum_{t=0}^{g}\binom{g}{t}e(L)+2\sum_{t=1}^{g-1}\binom{g-1}{t-1}\left[\sum_{j=1}^{g}e_1(L|I_j)\right]\right]\\
&= 2^{g-2}\left[e(L)+\sum_{j=1}^{g}e_1(L|I_j)\right].
\end{aligned}$$

□

We exhibit an interesting relationship between the multiplicity formula of the Rees algebra $\mathcal{R}(\mathbf{I})_{\mathcal{M}(\mathbf{I})}$ and that of the extended Rees algebra $\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}$.

Remark 3.11. Let $I_1, \ldots, I_g \subseteq \mathfrak{m}^2$ be ideals of positive height in $(R, \mathfrak{m})$. Then

$$e(\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}) = e(R) + \sum_{t=1}^{g}\sum_{1\leq i_1<\cdots<i_t\leq g} e(\mathcal{R}(I_{i_1},\ldots,I_{i_t})_{\mathcal{M}(I_{i_1},\ldots,I_{i_t})}).$$

Remark 3.12. The multiplicity formula characterizes the minimal multiplicity of $\mathcal{B}(\mathbf{I})_{\mathcal{N}(\mathbf{I})}$ (see [3]).

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